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Then 1 27 125 343 729....$(2n-1)^3$=given series.
26 98 218 386...............=first order of differences.
72 120 168......................=second order of differences.
48 48.........................=third order of differences.
0...............................=fourth order of differences.

Therefore $a=1$, $a_1=26$, $a_2=72$, $a_3=48$, $a_4=0$.

Substituting these values in the general formula, we obtain

$$Sum=n+13n(n-1)+12n(n-1)(n-2)+2n(n-1)(n-2)(n-3)$$
$$=2n^4-n^2=n^2(2n^2-1)$$

=*the sum of the cubes of the first n odd numbers.*

When $n=4$, $n^2(2n^2-1)=496$.

When $n=5$, $n^2(2n^2-1)=1225=\square$.

COROLLARIES BY M. A. GRUBER.

COROLLARY 1. In a similar manner we find $2^3+4^3+6^3....+(2n)^3=2n^2(n+1)^2$=*the sum of the cubes of the first n even numbers.*

COROLLARY 2. By a similar process we obtain $1+2^3+3^3+4^3+....+n^3=\left[\frac{n(n+1)}{2}\right]^2$; or *the sum of the cubes of the first n natural numbers is equal to the square of the sum of the numbers.* For $1+2+3+4+....+n=\frac{n(n+1)}{2}$.

COROLLARY 3. $1+3^3+5^3+... +(2n-1)^3=n^2(2n^2-1)=\square$, when $n=1$, 5, 29, 169, 985, 5741, etc., or when n=the integral hypothenuse of a right triangle whose legs are consecutive integers. [See AMERICAN MATHEMATICAL MONTHLY, Vol. IV., No. 1, pages 24-27].

COROLLARY 4. $2^3+4^3+6^3+....+(2n)^3$ can never be a square; for $2n^2(n+1)^2$ is *twice a square.*

GEOMETRY.

109. Proposed by CHARLES CARROLL CROSS, Libertytown, Md.

Two circles, radii in ratio 3:1, centers A and O_1 respectively, are drawn tangent externally to each other and internally to a given circle O, and on the same diameter; O_2 and O_2' are drawn tangent externally to O and internally to A and O_1; O_3 and O_3' are drawn tangent internally to O and externally to A and O_2; O_3 and O_3' are drawn tangent internally to O and externally to A and O_2, A and O_2', respectively, and so on. Prove O_4, O, O_4'; O_5, A, O_5'; O_9, A, O_3' and O_{10}, O, O_2' are collinear. [The letters apply to the centers of the circles].

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Construct the figure as indicated by the problem, and let B be the point of tangency of the circles A and O.

We will solve generally by taking $m:1$ as the ratio of the radii of circles A and O_1.

Put m=radius of circle A. Then 1=radius of circle O_1, $m+1$=radius of circle O, and $\frac{m(m+1)}{m(m+1)+(n-1)^2}$=radius of circle O_n, in which n=the number of the smaller circle, beginning with circle O_1.

(a). In $\triangle AO_nO$, taking AO as base,

$$AO_n=m+\frac{m(m+1)}{m(m+1)+(n-1)^2}=\frac{m\,[(m+1)^2+(n-1)^2]}{m(m+1)+(n-1)^2},$$

$$OO_n=m+1-\frac{m(m+1)}{m(m+1)+(n-1)^2}=\frac{(m+1)\,[m^2+(n-1)^2]}{m(m+1)+(n-1)^2},$$

$$AO=1,\ \text{area}=\frac{m(m+1)(n-1)}{m(m+1)+(n-1)^2},\ \text{and altitude}=\frac{2m(m+1)(n-1)}{m(m+1)+(n-1)^2}.$$

(b). We shall now find the value of n when centers O_n, O and O_r' are collinear.

Then $\angle O_r'OO_1=\angle O_nOB$; also OO_r' : sine $\angle O_r'OO_1=OO_n$: sine $\angle O_nOB$.

By substituting values as found in (a), and by using r and n in the proper places, we have

$$\frac{(m+1)\,[m^2+(r-1)^2]}{m(m+1)+(r-1)^3} : \frac{2m(m+1)(r-1)}{m(m+1)+(r-1)^2}$$

$$=\frac{(m+1)\,[m^2+(n-1)^2]}{m(m+1)+(n-1)^2} : \frac{2m(m+1)(n-1)}{m(m+1)+(n-1)^2}.$$

Whence $m^2+(r-1)^2 : r-1=m^2+(n-1)^2 : n-1$;

$m^2+(r-1)^2 : (n-1)^2-(r-1)^2=r-1 : n-r$;

$m^2+(r-1)^2 : n+r-2=r-1 : 1$.

From this we find $n=\frac{m^2}{r-1}+1$; also $r=\frac{m^2}{n-1}+1$.

When $r=n$, $(n-1)^2=m^2$; whence $n=m+1$.

$r=1$, $n=\infty$.

$r=2$, $n=m^2+1$.

$r=3$, $n=\frac{m^2}{2}+1$, m=*even* number.

$r=4$, $n=\dfrac{m^2}{3}+1$, $m=$multiple of 3.

$r=5$, $n=\dfrac{m^2}{4}+1$, $m=$*even* number.

etc., etc.

(c). We shall next find the value of n when centers O_n, A and O_r' are collinear.

Following a similar reasoning and substitution as in (b), we obtain

$$n=\frac{(m+1)^2}{r-1}+1\ ;\ \text{also}\ r=\frac{(m+1)^2}{n-1}+1.$$

When $r=n$, $(n-1)^2=(m+1)^2$; whence $n=m+2$.

$r=1$, $n=\infty$.

$r=2$, $n=(m+1)^2+1$.

$r=3$, $n=\dfrac{(m+1)^2}{2}+1$, $m=$*odd* number.

$r=4$, $n=\dfrac{(m+1)^2}{3}+1$, $m=3p-1$, as 2, 5, 8, 11, etc.

$r=5$, $n=\dfrac{(m+1)^2}{4}+1$, $m=$*odd* number.

etc., etc.

From (b) and (c) we obtain the following groups of collinear centers, when $m=3$: O_1, O, A, O_∞; O_2', O, O_{10}; O_2', A, O_{17}; O_3', A, O_9; O_4', O, O_4; and O_5', A, O_5.

It will be observed that where a center, in any of the above groups, has a *prime* mark. the mark may be transferred from the first to the last center of the group; as, for group O_2', O, O_{10}, we havs O_2, O, O_{10}'. This is evident from an inspection of the figure.

COROLLARY 1. The figure presents *three varieties of triangles*, which, in general forms, may be represented as AO_nO, $AO_{n+1}O_n$, and $OO_{n+1}O_n$, whose *areas are rational*. Area of triangle AO_nO is shown in (a).

COROLLARY 2. The curve passing through the centers of the smaller circles, ($i.\ e.$ through O_1, O_2, O_3 .... $O_\infty[=B]$ .... O_3', O_2', O_1), is an ellipse whose foci are A and O. For, $OO_1+AO_1=OO_2+AO_2=OO_n+AO_n=O_1B$.

The general values of the focal radii and ordinate of center O_r are shown in (a).

[NOTE. Mr. Charles C. Cross furnished me with the value $\dfrac{3}{13+n(n-2)}$ $=$radius of circle O_n, which I found correct according to his notation, and from which I deduced the general value $\dfrac{m(m+1)}{m(m+1)+(n-1)^2}$.]

Excellent solutions with diagrams were received from *G. B. M. ZERR*, *CHARLES C. CROSS*, and *J. SCHEFFER*.